Physics 523, General Relativity

Homework 5

Due Friday, 17th November 2006

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Problem 1

Let us consider a manifold with a torsion free connection $R(X,Y)$ which is not necessarily metric compatible. We are to prove that

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{1.1}$$

and the Bianchi identity

$$\nabla_X\big(R(X,Y)\big)V + \nabla_Y\big(R(Z,X)\big)V + \nabla_Z\big(R(X,Y)\big)V = 0. \tag{1.2}$$

The first identity is relatively simple to prove—it follows naturally from the Jacobi identity for the Lie derivative. Let us first prove the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{1.3}$$

Using the antisymmetry of the Lie bracket and our result from last homework problem 3, we have

$$[X,[Y,Z]] = \pounds_X[Y,Z] = -\pounds_{[Y,Z]}X = \pounds_Z\pounds_Y X - \pounds_Y\pounds_Z X = -[Z,[X,Y]] - [Y,[Z,X]].$$

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The condition of a connection being torsion free is that

$$\pounds_X Y = \nabla_X Y - \nabla_Y X. \tag{1.4}$$

Expanding the Lie brackets encountered in the statement of the Jacobi identity,

$$\begin{aligned}
0 =& \pounds_X\pounds_Y Z + \pounds_Y\pounds_Z X + \pounds_Z\pounds_X Y,\\
=& \pounds_X(\nabla_Y Z - \nabla_Z Y) + \pounds_Y(\nabla_Y X - \nabla_X Y) + \pounds_Z(\nabla_X Y - \nabla_Y X),\\
=& \nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y - \nabla_{[Y,Z]}X + \nabla_Y\nabla_Z X - \nabla_Y\nabla_X Z - \nabla_{[Z,X]}Y + \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X,Y]}Z,\\
=& \big(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\big)Z + \big(\nabla_Y\nabla_Z - \nabla_Z\nabla_Y - \nabla_{[Y,Z]}\big)X + \big(\nabla_Z\nabla_X - \nabla_X\nabla_Z - \nabla_{[Z,X]}\big)Y;
\end{aligned}$$

$$\therefore R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \tag{1.5}$$

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To prove the Bianchi identity, we will 'dirty' our expressions with explicit indices in hope of a quick solution. It is rather obvious to see that (1.2) is equivalent to the component expression

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0. \tag{1.6}$$

Worse than introducing components, let us use our (gauge) freedom to consider the Bianchi identity evaluated at a point p in spacetime in Riemann normal coordinates[1]. If we show that the Bianchi identity (1.6) holds in any particular coordinates at a point p, it necessarily must hold in any other coordinate system—and if p is arbitrary, then it follows that the Bianchi identity holds throughout spacetime.

Recall from lecture or elsewhere that Riemann normal coordinates at p are such that $\Gamma^a_{bc}(p) = 0$. This implies that the covariant derivative of the Riemann tensor is simply a normal derivative at p. Using the definition of $R^a{}_{bcd}$ in terms of the Christoffel symbols, we see at once that

$$R^a{}_{bcd;e}(p) + R^a{}_{bde;c}(p) + R^a{}_{bec;d}(p) = \Gamma^a_{bd,ce}(p) - \Gamma^a_{bc,de}(p) + \Gamma^a_{be,dc}(p) - \Gamma^a_{bd,ec}(p) + \Gamma^a_{bc,ed}(p) - \Gamma^a_{be,cd}(p);$$

$$\therefore R^a{}_{bcd;e}(p) + R^a{}_{bde;c}(p) + R^a{}_{bec;d}(p) = 0. \tag{1.7}$$

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[1]Riemann normal coordinates are constructed geometrically as follows: in a sufficiently small neighbourhood about p, every point can be reached by traversing a certain geodesic through p a certain distance. If we choose to define all families of geodesics through p using the same affine parameter λ then if we fix λ, there is a (smooth) bijection between tangent vectors in T_pM to points in the neighbourhood about p: the direction of $v \in T_pM$ tells the direction to the nearby points and its magnitude (for fixed λ) tells the distance to travel along the geodesic. Needless to say this construction does not require a metric.

Problem 2

We are to compute the Riemann tensor, the Ricci tensor, the Weyl tensor and the scalar curvature of a conformally-flat metric,

$$g_{ab}(x) = e^{2\Omega(x)}\eta_{ab}. \tag{2.1}$$

Using the definition of the Christoffel symbol with our metric above, we find

$$\begin{aligned}\Gamma^a_{bc} &= \frac{1}{2}g^{am}\left\{g_{am,b} + g_{bm,a} - g_{ab,m}\right\},\\ &= \frac{1}{2}e^{-2\Omega}\eta^{am}\left\{\eta_{bm}e^{2\Omega}\partial_c\Omega + \eta_{cm}e^{2\Omega}\partial_b\Omega - e^{2\Omega}\eta_{bc}\partial_m\Omega\right\},\\ \therefore\ \Gamma^a_{bc} &= \delta^a_b\partial_c\Omega + \delta^a_c\partial_b\Omega - \eta_{bc}\eta^{am}\partial_m\Omega.\end{aligned} \tag{2.2}$$

Using this together with the (definition of the) Riemann tensor's components

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^m_{bd}\Gamma^a_{cm} - \Gamma^m_{bc}\Gamma^a_{dm}, \tag{2.3}$$

we may compute directly[2],

$$\begin{aligned}R^a{}_{bcd} = &\ \delta^a_d\partial_c\partial_b\Omega - \eta_{bd}\eta^{am}\partial_c\partial_m\Omega - \delta^a_c\partial_b\partial_d\Omega + \eta_{bc}\eta^{am}\partial_d\partial_m\Omega - \delta^a_d\left(\partial_b\Omega\right)\left(\partial_c\Omega\right) + \eta_{bd}\eta^{am}(\partial_c\Omega)(\partial_m\Omega)\\ &- \delta^a_d(\partial_c\Omega)(\partial_b\Omega) - \delta^a_c(\partial_b\Omega)(\partial_b\Omega) + \eta_{bc}\delta^a_d\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) + \delta^a_c(\partial_b\Omega)(\partial_d\Omega) - \eta_{bc}\eta^{am}(\partial_d\Omega)(\partial_m\Omega)\\ &+ \delta^a_c(\partial_d\Omega)(\partial_b\Omega) + \delta^a_d(\partial_b\Omega)(\partial_c\Omega) - \eta_{bd}d^a_c\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) - \eta_{bd}\eta^{am}(\partial_c\Omega)(\partial_m\Omega) + \eta_{bd}\eta^{am}(\partial_c\Omega)(\partial_m\Omega)\\ =&\Big\{\delta^m_b\left(\delta^a_c\delta^n_d - \delta^a_d\delta^n_c\right) + \eta_{bd}\left(\eta^{an}\delta^m_c - \eta^{mn}\delta^a_c\right) + \eta_{bc}\left(\eta^{mn}\delta^a_d - \eta^{an}\delta^m_d\right)\Big\}\left(\partial_m\Omega\right)\left(\partial_n\Omega\right)\\ &+ \left(\delta^a_d\partial_c - \delta^a_c\partial_d\right)\partial_b\Omega + \eta^{am}\left(\eta_{bc}\partial_d\partial_m\Omega - \eta_{bd}\partial_c\partial_m\Omega\right).\end{aligned}$$

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It will be helpful to recast this into the form where all the indices are lowered. We can do this by acting with the metric tensor. Doing so we find,

$$\begin{aligned}e^{-2\Omega}R_{abcd} =&\Big\{\delta^m_b\left(\eta_{ac}\delta^n_d - \eta_{ad}\delta^n_c\right) + \eta_{bd}\left(\delta^n_a\delta^m_c - \eta_{ac}\eta^{mn}\right) + \eta_{bc}\left(\eta_{ad}\eta^{mn} - \delta^m_a\delta^n_d\right)\Big\}(\delta_m\Omega)(\delta_n\Omega)\\ &+ \eta_{ad}\partial_c\partial_b\Omega - \eta_{ac}\partial_d\partial_b\Omega + \eta_{bc}\partial_d\partial_a\Omega - \eta_{bd}\partial_c\partial_a\Omega,\\ =&\Big\{\eta_{ad}\delta^m_b\delta^n_c - \eta_{ac}\delta^m_b\delta^n_d + \eta_{bc}\delta^m_a\delta^n_d - \eta_{bd}\delta^m_a\delta^n_c\Big\}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \Big(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\Big)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega).\end{aligned} \tag{2.4}$$

Although we will not have any use for such frivolities, we can further compress this expression to

$$e^{-2\Omega}R_{abcd} = 4\delta^r_{[a}\,\delta^n_{b]}\delta^s_{[d}\,\delta^m_{c]}\eta_{rs}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \Big(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\Big)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega). \tag{2.5}$$

Now, we can then find the Ricci tensor by acting on equation (2.4) with g^{ac}. Letting D be the dimensionality of our manifold, we find

$$\begin{aligned}R_{bd} =&\Big\{\delta^m_d\delta^n_b - D\delta^m_d\delta^n_b + \delta^m_d\delta^n_b - \eta_{bd}\eta^{mn}\Big\}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \eta^{mn}\Big(\eta_{bd} - D\eta_{bd}\Big)(\partial_m\Omega)(\partial_n\Omega),\\ =&(2-D)\Big(\partial_b\partial_d\Omega - (\partial_b\Omega)(\partial_d\Omega)\Big) + (2-D)\eta_{bd}\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) - \eta_{bd}\eta^{mn}\partial_m\partial_n\Omega.\end{aligned} \tag{2.6}$$

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Lastly, contracting this, we find the scalar curvature,

$$\begin{aligned}e^{2\Omega}R =&(2-D)\eta^{mn}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + D(2-D)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) - D\eta^{mn}\partial_m\partial_n\Omega,\\ =&2(1-D)\eta^{mn}\partial_m\partial_n\Omega - (2-D)(1-D)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega).\end{aligned} \tag{2.7}$$

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[2]To be absolutely precise, there are two terms which manifestly cancel that appear when expanding this expression, which we have left out for typographical and aesthetic considerations.

All that remains for us to compute is the Weyl tensor. Any exposure to conformal geometry immediately tells us that the Weyl tensor vanishes. That is, that

$$R_{abcd} = \frac{1}{(D-2)}\Big(g_{ac}R_{db} + g_{db}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}\Big) - \frac{1}{(D-1)(D-2)}R\Big(g_{ac}g_{db} - g_{ad}g_{bc}\Big). \tag{2.8}$$

We will try as hard as possible to avoid actually computing the right hand side by expanding our expressions above. To show that the Weyl tensor vanishes, we must build R_{abcd} out of R_{bc}, R and the metric g_{ab}. This statement alone essentially gives us the expression at first glance.

The first important thing to notice is that R_{abcd} has no term proportional to $\eta^{mn}\partial_m\partial_n\Omega$ while both R_{ab} and R do. This means that if R_{abcd} can only be composed of linear combinations of R_{ab} and R which do not contain $\eta^{mn}\partial_m\partial_n\Omega$. Looking at expressions (2.4) and (2.6), we see that they can only appear in the combination

$$R_{bd} + \frac{e^{2\Omega}\eta_{bd}}{2(1-D)}R = R_{bd} + \frac{g_{bd}}{2(1-D)}R. \tag{2.9}$$

Any multiple of this combination will automatically have no $\eta^{mn}\partial_m\partial_n\Omega$ contribution.

Staring a bit more at equations (2.4) and (2.6), we notice that the first set of terms in (2.4) are all of the form $g_{ac}R_{bd}$. Indeed, we see that

$$\frac{1}{2-D}\Big\{\eta_{ad}R_{bc}-\eta_{ac}R_{bd}+\eta_{bc}R_{ad}-\eta_{bd}R_{ac}\Big\} = \Big\{\eta_{ad}\delta^m_b\delta^n_c-\eta_{ac}\delta^m_b\delta^n_d+\eta_{bc}\delta^m_a\delta^n_d-\eta_{bd}\delta^m_a\delta^n_c\Big\}\Big(\partial_m\partial_n\Omega-(\partial_m\Omega)(\partial_n\Omega)\Big)+\ldots . \tag{2.10}$$

Notice that multiplying both sides of the above equation by $e^{2\Omega}$ will convert all of the η_{ab}'s into g_{ab}'s[3] . This is all we need to construct the Riemann tensor from the Ricci tensor and scalar curvature: knowing the combination of Ricci tensors which gives part of the Riemann tensor, we can use (2.9) to determine the rest. Indeed, we see that

$$\begin{aligned}
C_{abcd} + R_{abcd} =& \frac{1}{2-D}\Big\{g_{ab}R_{bc} - g_{ad}R_{bd} + g_{bc}R_{ad} - g_{bd}R_{ac}\Big\} + \frac{R}{2(1-D)(2-D)}\Big(g_{ad}g_{bc} - g_{ac}g_{bd} + g_{bc}g_{ad} - g_{bd}g_{ac}\Big),\\
=& \frac{1}{D-2}\Big\{g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}\Big\} - \frac{R}{(D-1)(D-2)}\Big(g_{ac}g_{bd} - g_{ad}g_{bc}\Big),\\
=&\Big\{g_{ad}\delta^m_b\delta^n_c - g_{ac}\delta^m_b\delta^n_d + g_{bc}\delta^m_a\delta^n_d - g_{bd}\delta^m_a\delta^n_c\Big\}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \Big(g_{ad}g_{bc} - g_{ac}g_{bd}\Big)g^{mn}(\partial_m\Omega)(\partial_n\Omega),\\
=&R_{abcd};
\end{aligned}$$

$$\therefore C_{abcd} = 0. \tag{2.11}$$

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[3]The conversion from $\eta_{ab} \to g_{ab}$ is completely natural. The only possibly non-trivial step comes from the last term in the expression (2.4) for the Riemann tensor: bringing $e^{2\Omega}$ to the right hand side of (2.4), we have a term which has two lowered η_{ab}'s and one upper η_{ab}; now, $e^{2\Omega}\eta^{mn} = e^{4\Omega}g^{mn}$ and how these two factors of $e^{2\Omega}$ can be absorbed into the lowered η's as desired.

Problem 3

We are to show that if $\varphi(x)$ satisfies the flat-space, massless Klein-Gordon equation, then if $g_{ab} = e^{2\Omega(x)}\eta_{ab}$, the transformed field $e^{\beta\Omega(x)}\varphi(x) \equiv \varphi'(x)$ satisfies the equation

$$g^{ab}\varphi'_{;ab} - \alpha R\varphi' = 0, \tag{3.1}$$

for appropriate values of α and β—dependant on the spacetime dimension but independent of $\Omega(x)$.

Let us agree to call $\Box \equiv \eta^{ab}\partial_a\partial_b$. Then the flat-space Klein-Gordon equation is simply $\Box\varphi(x) = 0$. Recall the expression for the scalar curvature R in D spacetime dimensions for a metric which is conformally-related to the Minkowski metric (2.7):

$$R = 2(1-D)e^{-2\Omega}\Box\Omega - (2-D)(1-D)e^{-2\Omega}\eta^{mn}(\partial_m\Omega)(\partial_n\Omega). \tag{3.2}$$

We would like to explicitly state $g^{ab}\nabla_b\nabla_a$ in terms of $\Box$ and Ω. This can be done quite explicitly, recalling the Christoffel symbols for a conformally-flat spacetime(2.2),

$$\begin{aligned} g^{ab}\nabla_b\nabla_a &= g^{ab}\partial_a\partial_b - g^{ab}\Gamma^c_{ab}\partial_c, \\ &= e^{-2\Omega}\Big\{\Box - \eta^{ab}\Big(\delta^c_a(\partial_b\Omega)\partial_c + \delta^c_b(\partial_a\Omega)\partial_c - \eta_{ab}\eta^{cm}(\partial_m\Omega)\partial_c\Big)\Big\}, \\ &= e^{-2\Omega}\Big\{\Box - \eta^{cb}(\partial_b n\Omega)\partial_c - \eta^{ac}(\partial_a\Omega)\partial_c + D\eta^{cm}(\partial_m\Omega)\partial_c\Big\}, \\ &= e^{-2\Omega}\Big\{\Box - (D-2)\eta^{ab}(\partial_a\Omega)\partial_b\Big\}. \end{aligned}$$

Acting with $g^{ab}\nabla_b\nabla_a$ on φ' we find,

$$\begin{aligned} g^{ab}\nabla_b\nabla_a\varphi' &= e^{-2\Omega}\Big\{\Box\left(e^{\beta\Omega}\varphi\right) + (D-2)\eta^{ab}(\partial_a\Omega)\left(\partial_b\left(e^{\beta\Omega}\varphi\right)\right)\Big\}, \\ &= e^{-2\Omega}\Big\{\beta\varphi'\Box(\Omega) + \beta(\beta+D-2)\varphi'\eta^{ab}(\partial_a\Omega)(\partial_b\Omega) + 2\beta e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega) + (D-2)e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega)\Big\}. \end{aligned}$$

Although only one equation, if (3.1) is to hold for arbitrary $\Omega(x)$, there are actually three constraints implied by (3.1)—one for each functionally distinct contribution. Actually, we'll find that there are only two independent conditions—just enough to uniquely determine α and β.

First, notice that R does not contain any derivatives of $\varphi(x)$. Therefore equation (3.1) implies that

$$2\beta e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega) + (D-2)e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega) = 0, \tag{3.3}$$

arising from the $g^{ab}\nabla_b\nabla_a\varphi'$ term in (3.1). This obviously implies that

$$\therefore \beta = -\frac{D-2}{2}. \tag{3.4}$$

The next condition(s) come form matching the remaining two functionally distinct terms in (3.1), namely[4]

$$g^{ab}\nabla_b\nabla_a\varphi' - \alpha R\varphi' \propto \beta\Box\Omega + \beta(\beta+D-2)\eta^{ab}(\partial_a\Omega)(\partial_b\Omega) - 2\alpha(1-D)\Box\Omega + \alpha(D-2)(D-1)\eta^{ab}(\partial_a\Omega)(\partial_b\Omega). \tag{3.5}$$

Matching the corresponding terms, we see that

$$\alpha = \frac{\beta}{2(1-D)} \qquad \text{and} \qquad \alpha = \frac{-\beta(\beta+D-2)}{(D-2)(D-1)}. \tag{3.6}$$

We see that $\beta = \frac{1}{2}(D-2)$ is consistent with both of these—more concretely, any two of these three constraints is sufficient to imply the third. Therefore, we have shown that $\varphi' = e^{\beta\Omega}\varphi$ will satisfy the modified Klein-Gordon equation (3.1) for any $\Omega(x)$ if

$$\therefore \beta = \frac{2-D}{2} \qquad \text{and} \qquad \alpha = \frac{1}{4}\frac{D-2}{D-1}. \tag{3.7}$$

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[4]We are not including those pieces eliminated by the choice (3.4).